Radius of convergence of p-adic connections: an application to the p-adic Rolle theorem

by

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0 The p-adic Rolle theorem

We want to illustrate our global theory of the radius of convergence of a p-adic connection [1], by deducing from it a simple proof of a variant of the p-adic Rolle theorem [11, §2.4], [12, Prop. A.20], [7, Prop. 3.1]. The proof in itself follows from the most basic result on p-adic differential systems, namely the so-called *trivial estimate* for the radius of convergence of their solutions [5, p. 94]. In order to explain this implication, however, we need to review some basic notions and results of logarithmic differential calculus and of semistable reduction for curves, and to recall the main properties of the radius of convergence of a connection on a compact p-adic curve Y with singularities at Z [1]. We take this opportunity to clarify the relation between the global notion $\mathcal{R}_{\mathfrak{Y},\mathfrak{Z}}(x,(\mathcal{M},\nabla))$ of $(\mathfrak{Y},\mathfrak{Z})$-*normalized radius of convergence of* $(\mathcal{M},\nabla) \in \mathbf{MIC}((Y \setminus Z)/k)$ *at* $x \in Y \setminus Z$, introduced in [1], and the *intrinsic generic radius of convergence* $IR(\mathcal{M}_{(x)},\nabla)$ *of* $(\mathcal{M},\nabla)$ *at* x, for a point $x \in Y$ of Berkovich type 2 or 3, of Kedlaya [9, Def. 9.4.7]. The coincidence of the two notions when x is a point of type of the skeleton $\Gamma_{\mathfrak{Y},\mathfrak{Z}} \setminus Z$, is crucial in the conclusion of our proof.

Let $(k, |\ |)$ be a complete algebraically closed extension of $(\mathbb{Q}_p, |\ |_p)$, with $|p|_p = p^{-1}$, and let k° be the ring of integers of k. We denote by $\mathbb{P}$ (resp. $\mathbf{P}$) the algebraic (resp. Berkovich analytic) projective line over k. Let $\varphi(T) \in k(T)$ be a rational function of degree d. Then φ induces a finite flat map $\varphi : \mathbb{P} \to \mathbb{P}$, generically étale and locally free of rank d. Let B be the branch locus of φ, in the target $\mathbb{P}$, and let $Z = \varphi^{-1}(B)$, in the source $\mathbb{P}$. We prove the following variant of the p-adic Rolle theorem.

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Theorem 0.0.1. *Assume $D(0,1^-) = \{ x \in \mathbf{P} \mid |T(x)| < 1 \}$ does not intersect Z (viewed in $\mathbf{P}$). Then $\varphi : \mathbf{P} \to \mathbf{P}$ induces an injective map on any open disk of radius $p^{-\frac{1}{p-1}}$ centered at a k-rational point of $D(0,1^-)$.*

1 A change in viewpoint

We observe that φ restricts to an étale covering $\mathbb{P} \setminus Z \to \mathbb{P} \setminus B$ of degree d. Hence, $\Omega^1_{\mathbb{P}\setminus Z} = \varphi^*\Omega^1_{\mathbb{P}\setminus B}$ and $\varphi_*\Omega^1_{\mathbb{P}\setminus Z} = \varphi_*\mathcal{O}_{\mathbb{P}\setminus Z} \otimes_{\mathcal{O}_{\mathbb{P}\setminus B}} \Omega^1_{\mathbb{P}\setminus B}$, by the projection formula. The direct image

$$\varphi_*(d_{\mathbb{P}/k} : \mathcal{O}_{\mathbb{P}} \to \Omega^1_{\mathbb{P}}) \ , \quad \text{that is} \quad \varphi_*(d_{\mathbb{P}/k}) : \varphi_*\mathcal{O}_{\mathbb{P}} \to \varphi_*\mathcal{O}_{\mathbb{P}} \otimes (j_{\mathbb{P}\setminus B})_*\Omega^1_{\mathbb{P}\setminus B} \ ,$$

where $j_{\mathbb{P}\setminus B} : \mathbb{P} \setminus B \hookrightarrow \mathbb{P}$ denotes the open embedding, is then a connection on the locally free $\mathcal{O}_{\mathbb{P}}$-module $\mathcal{F} := \varphi_*\mathcal{O}_{\mathbb{P}}$ of rank d, with poles at B. We denote by $\mathbf{MIC}(\mathbb{P}(*B)/k)$ the tannakian category of such objects, so that

$$(1.0.1.1) \qquad (\mathcal{F}, \nabla_{\mathcal{F}}) := (\varphi_*\mathcal{O}_{\mathbb{P}}, \varphi_*(d_{\mathbb{P}/k})) \in \mathbf{MIC}(\mathbb{P}(*B)/k) \ ,$$

while, in the notation of [1], the analytification

$$(1.0.1.2) \qquad (\mathcal{F}^{\mathrm{an}}, \nabla_{\mathcal{F}^{\mathrm{an}}}) := (\mathcal{F}, \nabla_{\mathcal{F}})^{\mathrm{an}} \in \mathbf{MIC}(\mathbf{P}(*B)/k) \ .$$

It is clear that the k-vector space of k-analytic solutions of $(\mathcal{F}, \nabla_{\mathcal{F}})$ at any point z_0 of $\mathbb{P}(k) = \mathbf{P}(k)$ is spanned by the germs at z_0 of analytic sections of $\varphi : \mathbf{P} \to \mathbf{P}$. We are especially interested in the inverse image

$$(1.0.1.3) \qquad \varphi^*(\mathcal{F}, \nabla_{\mathcal{F}}) =: (\mathcal{E}, \nabla_{\mathcal{E}}) \in \mathbf{MIC}(\mathbb{P}(*Z)/k) \ .$$

The k-vector space of k-analytic solutions of $(\mathcal{E}, \nabla_{\mathcal{E}})$ at any point z_0 of $\mathbf{P}(k)$ is then spanned by the germs of analytic solutions $w(z)$ at $z = z_0$ of the algebraic equation $\varphi(w) = \varphi(z)$. The statement we are trying to prove follows if we can prove that for any k-rational point $z_0 \in D(0,1^-)$, the solutions of $(\mathcal{E}, \nabla_{\mathcal{E}})$ at z_0 converge in the open disk $D(z_0, (p^{-\frac{1}{p-1}})^-)$. This in turn is known to hold [5, p. 94] for the solutions of a system

$$(1.0.1.4) \qquad \Sigma : \frac{dY}{dT} = G\,Y \ ,$$

where G is a $d \times d$ matrix of analytic functions of T *bounded by 1 on* $D(0,1^-)$. Actually, we will also use a *Transfer Theorem* in a disk with no singularity, similar to [5, IV.5. A], which we now state and prove.

In the following definition, we suppose that the matrix G in (1.0.1.4) is a matrix of *analytic elements* [5, IV.4] on an open annulus

$$(1.0.1.5) \qquad C(0;\sigma,1) := \{x \in D(0,1^-) \,|\, \sigma < |T(x)| < 1 \} \ ,$$

for some $\sigma \in (0,1)$. Let $t_{0,1}$ be the boundary point of the disk $D(0,1^-)$, the so-called *Gauss point* of $\mathbf{P}$ (with respect to the formal coordinate T).

Definition 1.0.2. *The* generic radius of convergence $R(\Sigma)$ *of the system Σ of (1.0.1.4) is defined by extending the scalars from k to the* Gauss field $\mathscr{H}(t_{0,1})$*, so that the Gauss point $t_{0,1}$ determines a $\mathscr{H}(t_{0,1})$-rational point $t'_{0,1} \in \mathbf{P}\widehat{\otimes}_k\mathscr{H}(t_{0,1})$, such that $T(t'_{0,1}) = t_{0,1}$. Notice that the entries of G, hence the system 1.0.1.4, are analytic functions on the open disk of radius 1 in $\mathbf{P}\widehat{\otimes}_k\mathscr{H}(t_{0,1})$, centered at $t'_{0,1}$. Then $R(\Sigma)$ may be defined as the T-radius of the maximal open disk around $t'_{0,1}$, of radius not exceeding 1, on which all solutions of Σ converge.*

The number $R(\Sigma)$ is computed as follows. We first iterate (1.0.1.4) into

$$\frac{1}{n!}\,\frac{d^n\,Y}{(d\,T)^n} = G_{[n]}\,Y \ , \tag{1.0.2.1}$$

and then

$$R(\Sigma) = \min(1, \liminf_{i\to\infty} |G_{[i]}(t_{0,1})|^{-1/i}) \in (0,1] \ , \tag{1.0.2.2}$$

where the absolute value of a matrix is the maximum of the absolute values of its entries.

Theorem 1.0.3. (Transfer Theorem) *Any solution of Σ at any k-rational point $x \in D(0,1^-)$ converges in a disk of T-radius $R(\Sigma)$ around x.*

Proof. We have $|G_{[i]}(x)| \leq |G_{[i]}(t_{0,1})|$, and a solution matrix of Σ at x is given by $Y_x(T) = \sum_{n=0}^{\infty} G_{[i]}(x)(T-T(x))^n$. So, $Y_x(T)$ converges for $|T-T(x)| < R(\Sigma)$. □

The generic radius of convergence of (1.0.1.4) is bounded below as follows [5, p. 94].

Proposition 1.0.4. (Trivial Estimate)

$$R(\Sigma) \geq |G(t_{0,1})|^{-1}\,p^{-\frac{1}{p-1}} \ .$$

Our main task is therefore to prove the bound $|G(t_{0,1})| \leq 1$, for the matrix G expressing the connection $(\mathcal{E}, \nabla_{\mathcal{E}})$ in terms of a basis of sections of $\mathcal{E}$ on a Zariski neighborhood of $D(0,1^-)$. This amounts to proving that $(\mathcal{E}, \nabla_{\mathcal{E}})$ may be extended to a connection on a suitable integral model of $\mathbb{P}$.

Remark 1.0.5. Notice that the algebraic equation for w as a function of z, $\varphi(z+w) = \varphi(z)$ coincides with the equation $A_{\varphi}(z,w) = 0$ studied in section 2 of Faber's paper [7].

From now on we write $\varphi : Y \to X$ for $\varphi : \mathbb{P} \to \mathbb{P}$ (resp. $\varphi : Y^{\mathrm{an}} \to X^{\mathrm{an}}$ for $\varphi : \mathbf{P} \to \mathbf{P}$), in order to keep the distinction between source and target space (actually all of our discussion extends to any morphism of smooth k-projective curves $\varphi : Y \to X$). A more precise version of the previous statement is obtained if we view the pairs (Y,Z), (X,B) as smooth log-schemes over the log-field $(k, k^{\times})$ [8]. The rational function φ induces in fact a finite étale morphism $\varphi : (Y,Z) \to (X,B)$, locally free of degree d, and formula 1.0.1.1 admits the refinement

$$\nabla_{\mathcal{F}} : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^1_X(\log B) \ , \tag{1.0.5.1}$$

which shows that the natural X/k-connection with poles along B on the locally free $\mathcal{O}_X$-module of rank d, $\mathcal{F} = \varphi_* \mathcal{O}_Y$, admits logarithmic singularities along B. Similarly for

$$\nabla_{\mathcal{E}} : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_Y(\log Z) \ . \tag{1.0.5.2}$$

We point out once again that the same statements hold for $Y^{\mathrm{an}}/X^{\mathrm{an}}$.

2 Semistable models

In order to understand the integrality properties of $(\mathcal{E}, \nabla)$, we discuss continuation of φ to a morphism of strictly semistable models of Y and X. In this section, we prefer to assume that $(k, |\ |)$ is a complete *discretely valued* extension of $(\mathbb{Q}_p, |\ |_p)$, and we denote by R its ring of integers, which is then a DVR. The reason is that certain definitions and statements on semistable models and logarithmic structures are better known under these assumptions.

In practice, the new k will be any sufficiently big subfield of the original field k, finite over the closure of the field generated by the coefficients of φ.

From standard GAGA arguments and the algebraization of formal morphisms [6, Thm. 5.4.5], we have

Lemma 2.0.6. *Let Y be a smooth projective k-algebraic curve and let Y^{an} be the associated compact k-analytic curve. The completion functor $\mathfrak{Z} \mapsto \widehat{\mathfrak{Z}} := \widehat{\mathfrak{Z}_{/\mathfrak{Z}_s}}$ induces equivalences of categories*

$$\mathcal{PSS}(Y) \xrightarrow{\sim} \mathcal{FS}(Y^{\mathrm{an}}) \ , \tag{2.0.6.1}$$

between the category of proper (resp. strictly) semistable R-models of Y and the category of proper (resp. strictly) semistable formal R-models of Y^{an}.

The finite flat map of k-schemes $\varphi : Y \to X$ restricts to an étale covering $Y \setminus Z \to X \setminus B$ of degree d.

Lemma 2.0.7. *After passing to a finite extension of k, the map $\varphi : Y \to X$ admits a continuation to a proper surjective morphism of proper strictly semistable R-schemes $\Phi : \mathfrak{Y} \to \mathfrak{X}$ such that $\varphi = \Phi_\eta : \mathfrak{Y}_\eta \to \mathfrak{X}_\eta$.*

Proof. Granting lemma 2.0.6, we may analytify the morphism φ into $\varphi^{\mathrm{an}} : Y^{\mathrm{an}} \to X^{\mathrm{an}}$ and prove the corresponding statement for formal R-models. The existence of a continuation of φ^{an} to a morphism of admissible formal models follows from [4, Cor. 5.10 (a)]. Properness follows from [10, Lemma 2.6]. For surjectivity, one first uses flatness of φ (hence of φ^{an}) and [4, Cor. 5.10 (c)] to show that φ^{an} continues to a proper flat, hence surjective, morphism $\Phi' : \mathfrak{Y}' \to \mathfrak{X}'$ of admissible models of Y^{an}, X^{an}, respectively. Then, one makes an admissible blow-up $\mathfrak{X} \to \mathfrak{X}'$ so that $\mathfrak{X}$ is strictly semistable. One then applies base-change by $\mathfrak{X} \to \mathfrak{X}'$ to $\Phi' : \mathfrak{Y}' \to \mathfrak{X}'$, and obtain a proper flat morphism $\Phi'' : \mathfrak{Y}'' \to \mathfrak{X}$. Finally one finds an admissible blow-up $\mathfrak{Y} \to \mathfrak{Y}''$ so that $\mathfrak{Y}$ is strictly semistable. The composite morphism Φ has the required properties. ☐

Lemma 2.0.8. *By further admissible blowing-up of $\mathfrak{X}$ and $\mathfrak{Y}$, we may assume that the divisors $Z \subset Y$ and $B \subset X$ extend to divisors $\mathfrak{Z} \subset \mathfrak{Y}$ and $\mathfrak{B} \subset \mathfrak{X}$, respectively, both étale over R, and such that $\mathfrak{Y}_s \cup \mathfrak{Z}$ (resp. $\mathfrak{X}_s \cup \mathfrak{B}$) be a divisor with normal crossings in the regular R-scheme $\mathfrak{Y}$ (resp. $\mathfrak{X}$).*

Proof. Again by GAGA and [6, Thm. 5.4.5], we may deal with formal strictly semistable models of $\mathfrak{X}$ of X^{an} and $\mathfrak{Y}$ of Y^{an}, and with $\Phi : \mathfrak{Y} \to \mathfrak{X}$. A divisor $\mathfrak{Z}$ (resp. $\mathfrak{B}$) exists as soon as the points of Z (resp. of B) are k-rational of degree 1 lying in distinct residue classes of the analytic generic fiber of some smooth components of $\mathfrak{Y}$ (resp. $\mathfrak{X}$). Clearly, this condition may be achieved by admissible blowing-up. The divisor $\mathfrak{Z}$ (resp. $\mathfrak{B}$) is then given by the residue classes of smooth components of $\mathfrak{Y}$ (resp. $\mathfrak{X}$) containing one of the points of Z (resp. B). ☐

As a matter of notation, we recall that there is a canonical *specialization map*

$$\mathrm{sp}_{\mathfrak{Y}} : Y^{\mathrm{an}} = (\mathfrak{Y}_\eta)^{\mathrm{an}} = (\widehat{\mathfrak{Y}_{/\mathfrak{Y}_s}})_\eta \to \mathfrak{Y}_s \ , \tag{2.0.8.1}$$

($\mathfrak{Y}$ is proper) which may be viewed as a morphism of G-ringed spaces

$$\mathrm{sp}_{\mathfrak{Y}} : Y^{\mathrm{an}} \to \widehat{\mathfrak{Y}_{/\mathfrak{Y}_s}} \ . \tag{2.0.8.2}$$

We now fix an extension $\Phi : \mathfrak{Y} \to \mathfrak{X}$ of $\varphi : Y \to X$, and the divisors $\mathfrak{Z} \subset \mathfrak{Y}$ and $\mathfrak{B} \subset \mathfrak{X}$, as in lemmas 2.0.7 and 2.0.8. We refer now to [8] for the definitions and the differential

constructions related to log-schemes. We denote by $(\mathfrak{Y}, \mathfrak{Y}_s \cup \mathfrak{Z})$ (resp. $(\mathfrak{X}, \mathfrak{X}_s \cup \mathfrak{B})$), the smooth log-schemes over the log-ring $(R, R^\times)$. Since

$$\Phi(\mathfrak{Y} \setminus (\mathfrak{Y}_s \cup \mathfrak{Z})) \subset \mathfrak{X} \setminus (\mathfrak{X}_s \cup \mathfrak{B}) ,$$

the map Φ induces a proper étale morphism of smooth log-schemes over $(R, R^\times)$,

$$\Phi : (\mathfrak{Y}, \mathfrak{Y}_s \cup \mathfrak{Z}) \to (\mathfrak{X}, \mathfrak{X}_s \cup \mathfrak{B}) .$$

We will set

$$\mathfrak{F} := \Phi_* \mathcal{O}_{\mathfrak{Y}} \quad , \quad \mathfrak{E} := \Phi^* \Phi_* \mathcal{O}_{\mathfrak{Y}} .$$

Then $\mathfrak{F}$ (resp. $\mathfrak{E}$) is a coherent torsion-free $\mathcal{O}_{\mathfrak{X}}$-module (resp. $\mathcal{O}_{\mathfrak{Y}}$-module) such that $\mathfrak{F}_\eta = \mathcal{F}$ (resp. $\mathfrak{E}_\eta = \mathcal{E}$). As in the case of k-schemes, we use the projection formula for Φ, $\mathcal{O}_{\mathfrak{Y}}$ and $\Omega^1_{(\mathfrak{Y}, \mathfrak{Y}_s \cup \mathfrak{Z})/(R,R^\times)} = \Phi^* \Omega^1_{(\mathfrak{X}, \mathfrak{X}_s \cup \mathfrak{B})/(R,R^\times)}$, to obtain a logarithmic $(\mathfrak{Y}, \mathfrak{Y}_s \cup \mathfrak{Z})/(R, R^\times)$-connection

(2.0.8.3)

$$\Phi_*(d_{(\mathfrak{Y}, \mathfrak{Y}_s \cup \mathfrak{Z})/(R,R^\times)} : \mathcal{O}_{\mathfrak{Y}} \to \Omega^1_{(\mathfrak{Y}, \mathfrak{Y}_s \cup \mathfrak{Z})/(R,R^\times)}) =: (\nabla_{\mathfrak{F}} : \mathfrak{F} \to \mathfrak{F} \otimes_{\mathcal{O}_{\mathfrak{X}}} \Omega^1_{(\mathfrak{X}, \mathfrak{X}_s \cup \mathfrak{B})/(R,R^\times)}) ,$$

whose generic fiber is the connection (1.0.5.1). Then, by pull-back, we get a logarithmic $(\mathfrak{Y}, \mathfrak{Y}_s \cup \mathfrak{Z})/(R, R^\times)$-connection

$$\nabla_{\mathfrak{E}} : \mathfrak{E} \to \mathfrak{E} \otimes \Omega^1_{(\mathfrak{Y}, \mathfrak{Y}_s \cup \mathfrak{Z})/(R,R^\times)} , \tag{2.0.8.4}$$

whose generic fiber is the connection (1.0.5.2).

Proposition 2.0.9. *Let $\xi_{\mathcal{C}}$ be the generic point of a smooth component $\mathcal{C}$ of $\mathfrak{Y}_s$. The local ring $\mathcal{O}_{\mathfrak{Y}, \xi_{\mathcal{C}}}$ of $\xi_{\mathcal{C}}$ is a DVR with quotient field the function field $\kappa(Y)$ of Y and $\mathfrak{E}_{\xi_{\mathcal{C}}}$ (resp. $(\Omega^1_{\mathfrak{Y}/R})_{\xi_{\mathcal{C}}} = (\Omega^1_{(\mathfrak{Y}, \mathfrak{Y}_s \cup \mathfrak{Z})/(R,R^\times)})_{\xi_{\mathcal{C}}}$) is a free $\mathcal{O}_{\mathfrak{Y}, \xi_{\mathcal{C}}}$-module of rank d (resp. 1).*

A similar result holds of course for the $\mathcal{O}_{\mathfrak{X}}$-modules $\mathfrak{F}$ and $\Omega^1_{(\mathfrak{X}, \mathfrak{X}_s \cup \mathfrak{B})/(R,R^\times)}$, for any component of $\mathfrak{X}_s$. We have thus provided, at least in a neighborhood of $\xi_{\mathcal{C}}$, the integral structure on (1.0.5.2) which was needed in order to bound from below by proposition 1.0.4 the radius of convergence of solutions.

3 Graphs and radius of convergence

In this last section, it is convenient to resume the original assumption that $(k, |\ |)$ is a complete and algebraically closed extension of $(\mathbb{Q}_p |\ |_p)$. We recall from [1] that to the pair $(\mathfrak{Y}, \mathfrak{Z})$ we can associate a subgraph $\Gamma_{(\mathfrak{Y}, \mathfrak{Z})}$ of the profinite graph $Y^{\rm an}$, equipped with a continuous retraction $\tau_{(\mathfrak{Y}, \mathfrak{Z})} : Y^{\rm an} \to \Gamma_{(\mathfrak{Y}, \mathfrak{Z})}$. (We are extending the graph $\Gamma_{(\mathfrak{Y}, \mathfrak{Z})}$ of [1] to include the points of $Z \subset Y(k)$ as vertices "at infinite distance" and the retraction $\tau_{(\mathfrak{Y}, \mathfrak{Z})} : Y^{\rm an} \to \Gamma_{(\mathfrak{Y}, \mathfrak{Z})}$ by $\tau_{(\mathfrak{Y}, \mathfrak{Z})}(z) = z$, for any $z \in Z$.) The fibers of the retraction $\tau_{(\mathfrak{Y}, \mathfrak{Z})}$ are the closures in $Y^{\rm an}$ of the maximal open disks contained in $Y^{\rm an} \setminus \Gamma_{(\mathfrak{Y}, \mathfrak{Z})}$ and the singletons consisting of points of Z. Any such maximal open disk D contains at least a k-rational point $x \in Y(k)$; we define $D =: D_{(\mathfrak{Y}, \mathfrak{Z})}(x, 1^-)$. If the boundary point $\tau_{(\mathfrak{Y}, \mathfrak{Z})}(x) \in \Gamma_{(\mathfrak{Y}, \mathfrak{Z})}$ of $D_{(\mathfrak{Y}, \mathfrak{Z})}(x, 1^-)$ is of Berkovich type 2, $D_{(\mathfrak{Y}, \mathfrak{Z})}(x, 1^-)$ is isomorphic as a k-analytic curve to the standard open k-disk in $\mathbf{P}$, $D(0, 1^-)$, via a $(\mathfrak{Y}, \mathfrak{Z})$-*normalized coordinate at* x. Given any object $(\mathcal{M}, \nabla)$ of $\mathbf{MIC}((Y^{\rm an} \setminus Z)/k)$, and any $x \in X(k) \setminus Z$, we can define, as in [1], the $(\mathfrak{Y}, \mathfrak{Z})$-*normalized radius of convergence of* $(\mathcal{M}, \nabla)$ *at* x, $\mathcal{R}_{(\mathfrak{Y}, \mathfrak{Z})}(x, (\mathcal{M}, \nabla))$, as the radius, measured in $(\mathfrak{Y}, \mathfrak{Z})$-normalized coordinate at x, of the maximal open disk E centered at x and contained in $Y^{\rm an} \setminus \Gamma_{(\mathfrak{Y}, \mathfrak{Z})}$, such that $(\mathcal{E}, \nabla)_{|E}$ is a free $\mathcal{O}_E$-module of finite rank, equipped with the trivial connection. We can also extend the definition of $\mathcal{R}_{(\mathfrak{Y}, \mathfrak{Z})}(x, (\mathcal{M}, \nabla))$ to the case when $x \in Y^{\rm an} \setminus Z$ is not necessarily k-rational. In full generality, let K/k be

a completely valued field extension, let $Y_K^{\mathrm{an}} = Y^{\mathrm{an}}\widehat{\otimes}_k K$ and let $\pi_{K/k} : Y_K^{\mathrm{an}} \to Y^{\mathrm{an}}$, be the projection. Then there is a canonical functor *change of field of constants by* K/k

$$
\begin{array}{rrcl}
\pi^*_{K/k}: & \mathbf{MIC}((Y^{\mathrm{an}} \setminus Z)/k) & \to & \mathbf{MIC}((Y_K^{\mathrm{an}} \setminus Z)/K) \\
& (\mathcal{M}, \nabla) & \mapsto & \pi^*_{K/k}(\mathcal{M}, \nabla)\ .
\end{array} \tag{3.0.9.1}
$$

So, let $x \in Y^{\mathrm{an}} \setminus Z$, not necessarily k-rational. As in [1], we change the field of constants by $\mathscr{H}(x)/k$, and pick (canonically) a $\mathscr{H}(x)$-rational point $x' \in Y^{\mathrm{an}}\widehat{\otimes}_k\mathscr{H}(x)$ above x. We then set

$$
\mathcal{R}_{(\mathfrak{Y},\mathfrak{Z})}(x, (\mathcal{M}, \nabla)) := \mathcal{R}_{(\mathfrak{Y}\widehat{\otimes}_{k^\circ}\mathscr{H}(x)^\circ,\mathfrak{Z}\widehat{\otimes}_{k^\circ}\mathscr{H}(x)^\circ)}(x', \pi^*_{\mathscr{H}(x)/k}(\mathcal{M}, \nabla))\ . \tag{3.0.9.2}
$$

This definition is compatible with any change of the field of constants by any K/k in the sense that, for any K/k and any $y \in Y_K^{\mathrm{an}} \setminus Z$,

$$
\mathcal{R}_{(\mathfrak{Y}\widehat{\otimes}_{k^\circ}K^\circ,\mathfrak{Z}\widehat{\otimes}_{k^\circ}K^\circ)}(y, \pi^*_{K/k}(\mathcal{M}, \nabla)) = \mathcal{R}_{(\mathfrak{Y},\mathfrak{Z})}(\pi_{K/k}(y), (\mathcal{M}, \nabla))\ . \tag{3.0.9.3}
$$

The function $x \mapsto \mathcal{R}_{(\mathfrak{Y},\mathfrak{Z})}(x, (\mathcal{M}, \nabla))$ is conjectured to be continuous on $Y^{\mathrm{an}} \setminus Z$, for any $(\mathcal{M}, \nabla) \in \mathbf{MIC}((Y^{\mathrm{an}} \setminus Z)/k)$. This conjecture was proven in [1] under the assumption that $\mathcal{M}$ extends to a locally free coherent $\mathcal{O}_{\widehat{\mathfrak{Y}/\mathfrak{Y}_s}}$-module. We do not need this result in this discussion, nor could it directly be applied to the analytification of $(\mathcal{E}, \nabla_{\mathcal{E}})$, since the coherent $\mathcal{O}_{\mathfrak{Y}}$-module $\mathfrak{E}$ is not locally free, in general. But some comments are in order to explain the difference between our radius of convergence $\mathcal{R}_{(\mathfrak{Y},\mathfrak{Z})}(x, (\mathcal{M}, \nabla))$ and the *intrinsic radius of convergence* $IR(\mathcal{M}_{(x)}, \nabla)$ of

$$
(\mathcal{M}_{(x)}, \nabla) := (\mathcal{M}, \nabla)_x \otimes_{\mathcal{O}_{Y^{\mathrm{an}},x}} \mathscr{H}(x)\ , \tag{3.0.9.4}
$$

for $x \in Y^{\mathrm{an}}$ of Berkovich type 2 or 3, of Kedlaya [9, Def. 9.4.7]. Here $\mathcal{O}_{Y^{\mathrm{an}},x} = \kappa(x)$ is a valued field [3, 2.1], $(\mathcal{M}, \nabla)_x$ is a $\kappa(x)/k$-differential module and $(\mathcal{M}_{(x)}, \nabla)$ is its completion.[1] Both definition go back to Dwork and Robba; the latter was refined by Christol-Dwork and used by Christol-Mebkhout and André. We will show that two notions coincide at the points $x \in \Gamma_{(\mathfrak{Y},\mathfrak{Z})} \setminus Z$.

Let us shortly review, in our own words, the definition of $IR(\mathcal{M}_{(x)}, \nabla)$, taken from [9, Chap. 9]. A point $x \in \mathbf{P}$ of type 2 (resp. 3) is the point $t_{a,\rho}$ at the boundary of the open disk $D(a, \rho^-)$, for $a \in k$ and $\rho > 0$ in $|k|$ (resp. in $\mathbb{R} \setminus |k|$). One defines [9, Def. 9.4.1] $F_{a,\rho} = \mathscr{H}(x)$, as the completion of $k(T)$ under the absolute value

$$
f(T) \mapsto |f|_{a,\rho} := |f(t_{a,\rho})|\ .
$$

Let $\mathcal{L}_k(F_{a,\rho})$ be the k-Banach algebra of bounded k-linear endomorphisms of the k-Banach algebra $F_{a,\rho}$, equipped with the operator norm. We still denote the operator norm by $|\ |_{a,\rho}$. Then $\frac{d}{dT}$ extends by continuity to a k-derivation of $F_{a,\rho}$, and

$$
|\frac{d}{dT}|_{a,\rho} = \rho^{-1}\ , \tag{3.0.9.5}
$$

as an element of $\mathcal{L}_k(F_{a,\rho})$. For the spectral norm of $\frac{d}{dT} \in \mathcal{L}_k(F_{a,\rho})$, we have

$$
|\frac{d}{dT}|_{\mathrm{sp},a,\rho} = p^{-\frac{1}{p-1}}\,\rho^{-1}\ . \tag{3.0.9.6}
$$

[1] The reader should appreciate the difference between the operation $(\mathcal{M}, \nabla)) \mapsto (\mathcal{M}_{(x)}, \nabla)$, resulting in a $\mathscr{H}(x)/k$-differential module, and the change of field of constants by $\mathscr{H}(x)/k$, $(\mathcal{M}, \nabla) \mapsto \pi^*_{\mathscr{H}(x)/k}(\mathcal{M}, \nabla)$, resulting in an object of $\mathbf{MIC}((Y^{\mathrm{an}}_{\mathscr{H}(x)} \setminus Z)/\mathscr{H}(x))$.

Remark 3.0.10. Let $(F, |\ |_F)/(k, |\ |)$ be a complete extension field. Then $(F, |\ |_F)$ is a k-Banach algebra, and so is $\mathcal{L}_k(F)$, for the operator norm. Similarly, on a finite dimensional F-vector space M, all norms compatible with $|\ |_F$ are equivalent and define equivalent structures of k-Banach space on M. It will be understood in the following that any such M is given some norm of F-vector space, compatible with $|\ |_F$, and then $\mathcal{L}_k(M)$ is given the corresponding operator norm. The definitions will be independent of the choices made.

Under the previous assumptions $\mathcal{L}_k(F)$ (resp. $\mathcal{L}_k(M)$) will be regarded as an F-vector space via the *left* action, $(a\,L)(b) = a\,L(b)$, for $a, b \in F$ (resp. $a \in F$, $b \in M$) and $L \in \mathcal{L}_k(F)$ (resp. $\mathcal{L}_k(M)$) .

Definition 3.0.11. *A* complete differential field of dimension 1 over $(k, |\ |)$ *is a complete extension field* $(F, |\ |_F)/(k, |\ |)$ *such that the F-vector space $Der(F/k) \subset \mathcal{L}_k(F)$ of bounded k-linear derivations of F, is of dimension 1. A* based complete differential field (of dimension 1) over $(k, |\ |)$ *is a triple* $(F, |\ |_F, \partial)$ *where* $(F, |\ |_F)/(k, |\ |)$ *is a complete extension field and* $0 \neq \partial \in Der(F/k)$.

So, the pair (resp. the triple) $(F_{a,\rho}, |\ |_{a,\rho})$ (resp. $(F_{a,\rho}, |\ |_{a,\rho}, \frac{d}{dT})$) is a (resp. based) complete differential field of dimension 1 over $(k, |\ |)$.

Remark 3.0.12. Let $(F, |\ |_F)$ be a complete differential field of dimension 1 over $(k, |\ |)$. Then, for any F-basis ∂ of $Der(F/k)$ and for any $n \geq 0$, the F-vector subspace $Diff^n(F/k) \subset \mathcal{L}_k(F)$ of bounded k-linear differential operators of F of order $\leq n$, is freely generated by $\mathrm{id}_F, \partial, \ldots, \partial^n$.

Definition 3.0.13. *A* finite dimensional differential module over the complete differential field $(F, |\ |_F)$ *(of dimension 1 over $(k, |\ |)$) is a pair (M, ∇) consisting of a finite dimensional F-vector space M and of a k-linear bounded F-algebra homomorphism*

$$\nabla : Diff(F/k) \to \mathcal{L}_k(M) \ ,$$

such that

$$\nabla(\partial)(a\,m) = \partial(a)\,m + a\,\nabla(\partial)(m) \ ,$$

for any $\partial \in Der(F/k)$, $a \in F$ and $m \in M$. If we specify a generator ∂ of $Der(F/k)$ and the corresponding $\Delta = \nabla(\partial)$, we obtain the based *finite dimensional differential module* (M, Δ) *over the based complete differential field* $(F, |\ |_F, \partial)$.

Remark 3.0.14. Conversely, given a based finite dimensional differential module (M, Δ) over the based complete differential field $(F, |\ |_F, \partial)$, one defines (M, ∇) by setting

$$\nabla(\sum_{i=0}^{n} a_i\,\partial^i) = \sum_{i=0}^{n} a_i\,\Delta^i \ ,$$

for any n and any $a_0, \ldots, a_n \in F$. It is clear that ∇ is a bounded F-algebra homomorphism is a

$$\nabla : Diff(F/k) \to \mathcal{L}_k(M) \ .$$

Definition 3.0.15. *Let $(M, \nabla(\partial)) = (M, \Delta)$ be a nonzero finite dimensional based differential module over the based complete differential field $(F, |\ |_F, \partial)$. The* extrinsic radius of convergence *of (M, Δ) is*

$$R(M, \Delta) = p^{-\frac{1}{p-1}}\,|\Delta|_{\mathrm{sp}}^{-1} > 0 \ ,$$

where $|\Delta|_{\mathrm{sp}}$ is the spectral norm of Δ of the k-Banach algebra $\mathcal{L}_k(M)$.

Definition 3.0.16. *Let (M,∇) be a finite dimensional differential module over the complete differential field $(F,|\ |_F)$,*

The intrinsic radius of convergence *of (M,∇) is*

$$IR(M,\nabla) = R(M,\nabla(\partial))\, p^{\frac{1}{p-1}}\, |\partial|_{\mathrm{sp}} = |\partial|_{\mathrm{sp}}\, |\Delta|_{\mathrm{sp}}^{-1} \in (0,1]\ ,$$

for any non zero element $\partial \in Der(F/k)$.

The following proposition explains why $IR(M,\nabla)$ deserves the attribute *intrinsic*.

Proposition 3.0.17. *For any $n = 0,1,\dots,$ let $c_n \in \mathbb{R}_{>0}$ be the operator norm of the map of k-Banach spaces*

$$\nabla_n = \nabla_{|Diff^n(F/k)} : Diff^n(F/k) \to \mathcal{L}_k(M)\ .$$

Then

$$IR(M,\nabla) = \liminf_{n\to\infty} c_n^{-1/n}\ . \tag{3.0.17.1}$$

Proof. Essentially follows from [9, Prop. 6.3.1]. □

Corollary 3.0.18. *Let $(\mathcal{M},\nabla) \in$ $\mathbf{MIC}((Y^{\mathrm{an}}\setminus Z)/k)$, as before. Let $x \in Y^{\mathrm{an}}$ be a point of Berkovich type 2 or 3. Let D be any open disk in Y^{an} with boundary point x, let*

$$S : D \xrightarrow{\ \sim\ } D(0,1^-)$$

be a normalized coordinate on D and, for any $\sigma \in (0,1)$, let

$$C_\sigma := S^{-1}(C(0;\sigma,1)) \subset Y^{\mathrm{an}} \setminus Z\ .$$

For σ close to 1, we identify the restriction $(\mathcal{M},\nabla)_{|C_\sigma}$, via the coordinate S and the choice of a basis of sections of $\mathcal{M}$ in a neighborhood of x containing C_σ, with a differential system Σ of the form 1.0.1.4. Then

$$IR(\mathcal{M}_{(x)},\nabla) = R(\Sigma)\ . \tag{3.0.18.1}$$

Remark 3.0.19. In formula 3.0.17.1, no formal semistable model $\mathfrak{Y}$ of Y explicitly appears. Such a (smooth) model is hidden, however, in the absolute value corresponding to the point x of type 2 or 3. As explained in corollary 3.0.18, the normalization of measures at x of type 2 or 3 in this case varies with x and is obtained by taking as an open disk of radius 1, *any open disk with boundary point x.*

The disadvantage of the function $x \mapsto IR(\mathcal{M}_{(x)},\nabla)$ which describes the intrinsic radius of convergence of $\mathcal{M}$ at $x \in Y^{\mathrm{an}}$ of type 2 or 3, is that it cannot possibly be extended by continuity to $Y^{\mathrm{an}}\setminus Z$. In fact, for any point $x_0 \in Y(k)\setminus Z$, one obviously has

$$\lim_{x\to x_0} R((\mathcal{M},\nabla),x) = 1\ , \tag{3.0.19.1}$$

where the limit runs over the points x of type 2 or 3. But, $Y(k)\setminus Z$ is dense in $Y^{\mathrm{an}}\setminus Z$, so one would have $R((\mathcal{M},\nabla),x) = 1$ identically on $Y^{\mathrm{an}}\setminus Z$, which is obviously not always the case. The point in our definition of the function $x \mapsto \mathcal{R}_{(\mathfrak{Y},\mathfrak{Z})}(x,(\mathcal{M},\nabla))$ [1] is that

1. it interpolates the classical notion of radius of convergence, normalized by the choice of $(\mathfrak{Y},\mathfrak{Z})$;

2. it is compatible with extension of the ground field;

3. it coincides on the graph $\Gamma_{(\mathfrak{Y},\mathfrak{Z})}$ with intrinsic radius of convergence

$$IR(\mathcal{M}_{(x)},\nabla) = \mathcal{R}_{(\mathfrak{Y},\mathfrak{Z})}(x,(\mathcal{M},\nabla))\ ,\ \text{ if }\ x \in \Gamma_{(\mathfrak{Y},\mathfrak{Z})}\ . \tag{3.0.19.2}$$

The last property follows from remark 3.0.19.

4 Conclusion of the proof

We recall that we chose a suitable extension $\Phi : \mathfrak{Y} \to \mathfrak{X}$ of $f : Y \to X$. As for the mutual position of $D(0,1^-) \subset \mathbf{P} = Y^{\mathrm{an}}$ (we are here using the original coordinate T) and $\Gamma_{(\mathfrak{Y},\mathfrak{Z})}$, there are only two possibilities:

1. $D(0,1^-) \cap \Gamma_{(\mathfrak{Y},\mathfrak{Z})} = \emptyset$,
2. the boundary point $t_{0,1}$ of $D(0,1^-)$ in $\mathbf{P}$ belongs to the support of $\Gamma_{(\mathfrak{Y},\mathfrak{Z})}$. By a further blow-up, we may assume that it is a vertex.

Let now v be any vertex of $\Gamma_{(\mathfrak{Y},\mathfrak{Z})}$ of Berkovich type 2 (*i.e.* any vertex which is not in Z). Then $\mathrm{sp}_{\mathfrak{Y}}(v)$ is the generic point $\xi_{\mathcal{C}}$ of one smooth component $\mathcal{C}$ of the special fiber $\mathfrak{Y}_s$ (and $\mathrm{sp}_{\mathfrak{Y}}^{-1}(\{\xi_{\mathcal{C}}\}) = \{v\}$). We denote v by $\eta_{\mathcal{C}}$, called *the generic point of* $\mathcal{C}$ *in* Y^{an}, and denote by $(\mathscr{H}(\eta_{\mathcal{C}}), |\ |_{\eta_{\mathcal{C}}})$, the corresponding complete valued field. Now, by proposition 2.0.9, the fiber of the coherent torsion-free $\mathcal{O}_{\mathfrak{Y}}$-module $\mathfrak{E}$ at $\xi_{\mathcal{C}} \in \mathfrak{Y}$ is a free $\mathcal{O}_{\mathfrak{Y},\xi_{\mathcal{C}}}$-module $\mathfrak{E}_{\xi_{\mathcal{C}}}$ of finite rank d and the $\mathcal{O}_{\mathfrak{Y},\xi_{\mathcal{C}}}$-module $(\Omega^1_{(\mathfrak{Y},\mathfrak{Z})/(R,R^\times)})_{\xi_{\mathcal{C}}} = (\Omega^1_{\mathfrak{Y}/R})_{\xi_{\mathcal{C}}}$ is freely generated by $d_{\mathfrak{Y}/R}S$, for any parameter S of $\mathcal{O}_{\mathfrak{Y},\xi_{\mathcal{C}}}$. We apply the construction 3.0.9.4, to obtain the differential module $(M,\nabla) = (\mathcal{E}_{(\eta_{\mathcal{C}})}, \nabla)$ over the complete differential field $(\mathscr{H}(\eta_{\mathcal{C}}), |\ |_{\eta_{\mathcal{C}}})$ of dimension 1 over $(k, |\ |)$. Any free system of $\mathcal{O}_{\mathfrak{Y},\xi_{\mathcal{C}}}$-generators $\mathbf{e} = (e_1, \ldots, e_d)$ of $\mathfrak{E}_{\xi_{\mathcal{C}}}$ is at the same time a basis of the $\mathscr{H}(\eta_{\mathcal{C}})$-vector space M and a basis of sections of $\mathcal{E}^{\mathrm{an}}$ in a neighborhood of $\eta_{\mathcal{C}}$.

We have

$$\nabla_{\mathcal{E}}(\mathbf{e}) = -(\mathbf{e} \otimes d_{\mathfrak{Y}/R}S)\, G \ , \tag{4.0.19.3}$$

for a $d \times d$-matrix G with entries in $\mathcal{O}_{\mathfrak{Y},\xi_{\mathcal{C}}}$. It follows from the trivial estimate (1.0.4) that the local analytic solutions of the corresponding differential system

$$\Sigma : \frac{dY}{dS} = GY \ , \tag{4.0.19.4}$$

at $\eta_{\mathcal{C}}$ in the sense of [1], have S-radius of convergence

$$IR(M,\nabla) = \mathcal{R}_{(\mathfrak{Y},\mathfrak{Z})}(\eta_{\mathcal{C}}, (\mathcal{E}, \nabla_{\mathcal{E}})^{\mathrm{an}}) \geq p^{-\frac{1}{p-1}} \ . \tag{4.0.19.5}$$

Remark 4.0.20. The previous estimate means that, for any k-rational point z in the affinoid with good canonical reduction

$$C = \mathrm{sp}_{\mathfrak{Y}}^{-1}(\mathcal{C} \setminus (\mathcal{C} \cap (\mathfrak{Z}_s \cup \mathfrak{Y}_s^{\mathrm{sing}}))\ , \tag{4.0.20.1}$$

the maximal common S-radius of convergence of the solutions of (4.0.19.4) at z is $\geq p^{-\frac{1}{p-1}}$. Therefore, the radius of convergence of the solutions of (4.0.19.4) at z, expressed in the coordinate $T - T(z)$, is

$$T\text{-radius of convergence} \geq p^{-\frac{1}{p-1}} \times \text{ the } T\text{-radius of a residue class of } C \ . \tag{4.0.20.2}$$

In case 2, we apply these considerations to $\eta_{\mathcal{C}} = t_{0,1}$ and to $S = T$. By the main assumption of theorem 0.0.1, namely the fact that $D(0,1^-)$ does not contain any singularity of the connection $(\mathcal{E}, \nabla_{\mathcal{E}})$, we deduce that, with respect to some choice of a basis of rational sections of $\mathcal{E}$, the solutions of $(\mathcal{E}, \nabla_{\mathcal{E}})$ in $D(0,1^-)$ satisfy the system (1.0.1.4), where the matrix G has entries which are rational functions with no pole in $D(0,1^-)$. But the choice of a basis of sections of $\mathcal{E}$ does not affect the intrinsic radius of convergence $IR(M,\nabla) \geq p^{-\frac{1}{p-1}}$. We are then in the position to apply the transfer theorem 1.0.3 to get precisely the statement (0.0.1).

In case 1, the open disk $D(0,1^-)$ is contained in a unique maximal open disk D' contained in $Y\setminus\Gamma_{(\mathfrak{Y},\mathfrak{Z})}$ and the boundary point of D' belongs to the support of $\Gamma_{(\mathfrak{Y},\mathfrak{Z})}$. By an admissible blow-up, we make that boundary point a vertex of $\Gamma_{(\mathfrak{Y},\mathfrak{Z})}$, and we are again in the situation 2, with $D(0,1^-)$ replaced by the bigger open disk D'. So, the statement of theorem 0.0.1 holds *a fortiori*.

Our proof is complete.

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